Herd Design

Itai Arieli[*] Ronen Gradwohl[†] Rann Smorodinsky[‡]

Abstract

The classic herding model examines the asymptotic behavior of agents who observe their predecessors' actions as well as a private signal from an exogenous information structure. In this paper we introduce a self-interested sender into the model and study her problem of designing this information structure. If agents cannot observe each other the model reduces to Bayesian persuasion. However, when agents observe predecessors' actions, they may learn from them, potentially harming the sender. We identify necessary and sufficient conditions under which the sender can nevertheless obtain the same utility as when the agents are unable to observe each other.

[*]Faculty of Data and Decision Sciences, Technion, Israel. Email: iarieli@technion.ac.il. Arieli gratefully acknowledges support of Israel Science Foundation grant 2029464, Ministry of Science and Technology grant 19400214, and the Bernard M. Gordon Center for Systems Engineering at the Technion.

[†]Department of Economics and Business Administration, Ariel University, Israel. Email: roneng@ariel.ac.il. Gradwohl gratefully acknowledges the support of National Science Foundation award number 1718670.

[‡]Faculty of Data and Decision Sciences, Technion, Israel. Email: rann@ie.technion.ac.il. Smorodinsky gratefully acknowledges support of Israel Science Foundation grant 2646/22, United States-Israel Binational Science Foundation and National Science Foundation grant 2016734, German-Israeli Foundation grant I-1419-118.4/2017, Ministry of Science and Technology grant 19400214, Technion VPR grants, and the Bernard M. Gordon Center for Systems Engineering at the Technion.

1 Introduction

A striking insight from information economics is that rational agents who learn from others may fail to fully aggregate information. A hallmark example of such market failure is the classic observational learning model, in which agents act sequentially after observing their predecessors' actions as well as a signal from an exogenous information structure. The resulting market failure has been used to explain a variety of phenomena, ranging from IPO underpricing and microloan markets to software adoption and crowdfunding.[1]

In the observational learning model, the failure to aggregate information leads agents to herd on a potentially inferior action, with the particular action depending on the exogenous information structure. Where does this information structure come from? In all the examples above, the information is provided by one of the parties involved: The issuer of the IPO publishes a prospectus, the potential borrower posts credit information and personal attributes, and the software developer and innovator share product information. However, these parties are not disinterested; on the contrary, they have preferences over the resulting herd. The question, then, is to what extent they can determine the direction of the herd by exerting control over the provision of information.

We consider this question through the lens of Bayesian persuasion. Bayesian persuasion examines how an informed *sender* should share information in order to manipulate others to act in a way that benefits her. We suppose such a sender designs the information structure in the observational learning model, and ask, can the sender manipulate the crowd as well as she can manipulate any individual?

If agents cannot observe each other and their only source of information is the signal, then our model reduces to the standard Bayesian persuasion setting. However, when agents can observe each other they obtain additional information. We identify necessary and sufficient conditions under which the sender can nevertheless obtain the same utility as when the agents are unable to observe each other.

Our proof is constructive and provides an ε-optimal information structure. In contrast with the classic Bayesian persuasion result where, in a binary-state setting, two signals suffice for the sender to attain the optimal outcome, our sender requires a much richer set of signals.

[1] See Welch (1992), Zhang and Liu (2012), Duan et al. (2009), and Thies et al. (2016), respectively.

To further motivate our work consider the challenge of designing marketing strategies in online retail. In such settings, buyers obtain signals about products' qualities and characteristics from advertisements and product information shared by the seller. For instance, on Amazon.com, listed products contain brief descriptions, labels such as "Amazon's Choice" or "Small Business," and photographs. However, in addition to these signals, buyers typically also learn about the volume of previous purchases. When contemplating the purchase of a product, a buyer will use the signal and the information inferred from the volume of previous purchases to make a decision. Such observational learning can lead to herding (see, e.g., Chen et al., 2011; Tucker and Zhang, 2011).

A seller's marketing strategy in this market consists of designing the advertisements and provision of product information that make up buyers' signals.[2] If these signals were the buyers' only source of information, the seller would design them in a way that maximizes each buyer's probability of purchasing. When buyers have information about previous purchases, however, herding can lead to a reduction in sales. In this paper we identify conditions under which the seller can neutralize this mitigating effect of information about previous purchases, and can design signals that lead to the sales volume achievable without such information. Section 2 below illustrates our result in the context of this retail example.

1.1 Related Literature

Our work lies in the intersection of two rich and vibrant fields of research, herding (Bikhchandani et al., 1992; Banerjee, 1992) and Bayesian persuasion (Kamenica and Gentzkow, 2011). The primary insight of the herding literature is that market failure is possible and information need not be aggregated. This observation inspired a design question (Sgroi, 2002; Acemoglu et al., 2011; Bahar et al., 2020; Smith et al., 2020; Arieli et al., 2021): How can a benevolent designer control information transmission among agents to improve information aggregation and induce more efficient outcomes? Much of this work focuses on social networks where agents observe only a partial subset of previously active agents. We take an orthogonal perspective and focus on the optimal design of a self-motivated sender who has no control over the observability structure.

[2] See Kamenica and Gentzkow (2011) for a discussion of Bayesian persuasion as a model for the provision of product information.

As a first model we maintain the standard assumption that agents observe the actions taken by all past agents.

Our paper is also related to a line of work within the Bayesian persuasion literature which studies sequential information design (such as Au, 2015; Ely, 2017; Li and Norman, 2021; Lorecchio and Monte, forthcoming). The work of Lorecchio (2022) is most-closely related: He considers a binary-action observational learning model with a designer who provides agents with additional information, and derives conditions under which the designer prefers observational learning and under which she prefers to induce a herd from the start. In this line of work the design challenge is dynamic, and the resulting signaling scheme can change from one stage to another. In contrast, our sender commits to one information structure at the beginning, and each agent's signal is then drawn from that structure. The static information structure is canonical in the herding literature, and it fits the examples in the introduction well: For instance, the IPO prospectus and the retailer's posted photographs and product information often remain unchanged after publication. We note that, if our sender could commit to different structures for different agents, the problem would reduce to the standard Bayesian persuasion problem: The sender would commit to the optimal structure for the first agent, and subsequently reveal no further information.

Finally, our paper is related to work on Bayesian persuasion with informational spillovers between agents, such as Candogan (2020), Egorov and Sonin (2020), and Galperti and Perego (2020). The main difference is that these papers model spillovers through direct information transmission on a network, whereas in our paper spillovers occur indirectly through agents' actions.

2 Illustrative Examples

Recall that, in the Bayesian persuasion model, when the state of the world is binary, the sender's optimal information structure induces posterior beliefs that coincide with the two distributions corresponding to the concavification of the sender's utility function. Our main question in this paper reduces to the question of whether it is possible to design an information structure such that the *public belief*—the posterior implied by the sequence of agents' actions thus far—will converge to one of these two posteriors. We turn to two examples that illustrate the issues involved.

Example 1. A retailer sells identical copies of a product to many consumers. With probability $\mu = \frac{1}{5}$ the product is of high quality and with probability $\frac{4}{5}$ it is of low quality. Consumers' preferences are such that they have a positive expected utility from purchasing the product whenever their belief that the product is high quality is at least $T = \frac{1}{4}$.

Consumers arrive at the market sequentially, and each observes the purchase decisions made by prior consumers as well as a conditionally independent signal about the quality of the product. The design of the signal is done by the retailer, whose goal is to maximize sales.

From the herding literature we know that any such design will eventually result in all consumers, from some point on, choosing the same action. Thus, the retailer's goal is to design an information structure that maximizes the probability that consumers herd on the purchase action.

What information structure should the retailer choose? One possible structure is the fully revealing one, in which the quality of the product is fully revealed. In this case, consumers will purchase the product if it is revealed to be of high quality, which happens with probability $\mu = \frac{1}{5}$.

Can the retailer do better? One may be tempted to let the retailer commit to the optimal information structure from the single-receiver Bayesian persuasion problem (Kamenica and Gentzkow, 2011). This information structure contains two signals, $\{h, \ell\}$. If the product is high quality, then the realized signal is always h, and if it is low quality, then the signal is a randomization of h and ℓ. The randomization is such that, on signal h, the consumer's belief that the product is high quality is $T = \frac{1}{4}$, while on the opposite signal the belief is zero. This structure yields the receiver expected utility $\frac{4}{5}$ *from the first consumer*, which is optimal. However, in our sequential setting, this structure is no better than the fully revealing one: If a consumer chooses not to purchase, then all subsequent agents learn with certainty that the product is low quality. Thus, if the product is indeed low quality, then at some point some consumer will obtain signal ℓ, will choose not to purchase, and then all subsequent consumers will also refuse to purchase.

A simple way to increase the probability that consumers herd on the purchase decision is to choose an information structure that reveals the quality of the product with probability p and the opposite quality with probability $1-p$. Choose p so that a consumer with prior $\mu = \frac{1}{5}$ and signal h updates to posterior $T = \frac{1}{4}$ (and so makes a purchase). With this signal, the probability of herding on the purchase action is close to $\frac{52}{185} > \frac{1}{5}$, which is better than full revelation.

For intuition, observe that if the first consumer obtains signal ℓ and chooses not to purchase the product, then all subsequent retailers will also choose not to purchase. This is because, regardless of their signal, their belief remains below T, and so their choice does not reveal any new information to subsequent consumers. Alternatively, if the first two consumers obtain signal h and purchase the product, then all subsequent consumers will also make a purchase, regardless of their signals. Thus, eventually the belief about the quality of the product will either be the one induced by one ℓ signal or the one induced by two h signals, starting from the prior. The probability of each such belief must be such that the expected posterior is equal to the prior $\mu = \frac{1}{5}$. A simple calculation implies that the probability of a herd on the purchase decision is $\frac{52}{185}$.

We now turn back to our research question: Is it possible for the retailer to design an information structure that leads to expected utility equal to the concavification of her utility at the prior, namely, $\frac{4}{5}$? Recall that the threshold for buying is $T = \frac{1}{4}$, and so under the concavification the public belief must converge to one of the two posteriors, 0 and $\frac{1}{4}$.

For this example, our result is positive. In particular, for any $\varepsilon > 0$ it is possible to construct an information structure for which the public belief converges to either ε or $\frac{1}{4} + \varepsilon$. Under this information structure agents will herd on the no-buy action with probability close to $\frac{1}{5}$ and on the buy action with probability close to $\frac{4}{5}$, which delivers the retailer expected utility close to the optimal $\frac{4}{5}$.

In contrast with the binary information structure that is sufficient for this maximal utility in the Bayesian persuasion setting, the one we turn to describe is quite rich. In what follows we refer to signals that sway the agent's belief towards the high quality state as high signals and those that sway it the other way as low signals.

First, note that in order for the public belief converge to ε, it must be that if the public belief is greater than ε then for some signal the next agent will, nevertheless, take the buy action; for otherwise, learning will stop and the public belief will converge to a number greater than ε. On the other hand, this should not happen when the public

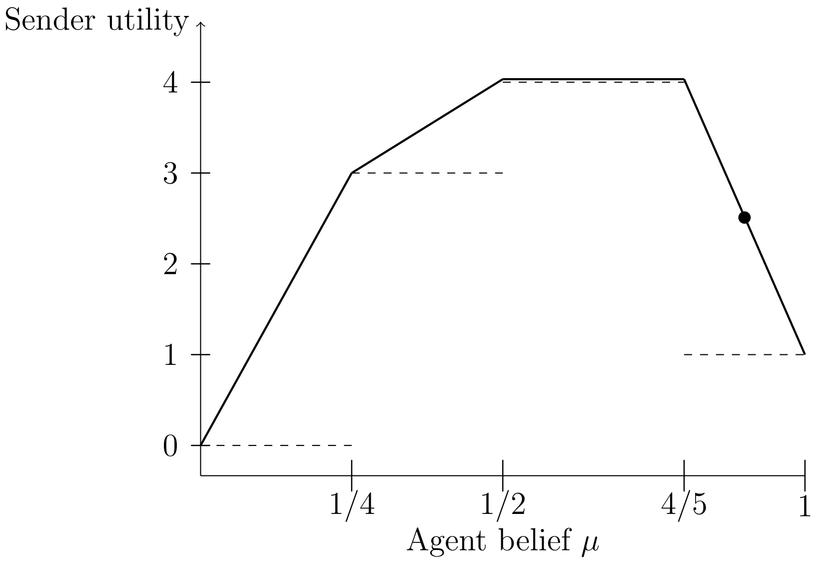

Figure 1: The dashed line depicts the sender's indirect utility as a function of the agent's (posterior) belief. The solid curve is the concavification of that utility function.

belief is ε or less. This suggests an upper bound on the high signals. Similarly, whenever the belief is higher than $\frac{1}{4}+\varepsilon$, no agent should take the no-buy action and sway the public belief away. This induces a lower bound on the low signals and implies that they sway the public belief downwards gently. Finally, in order for the belief to converge to $\frac{1}{4}+\varepsilon$ it must be the case that the belief never jumps beyond that value. This means that as long as the public belief is between ε and $\frac{1}{4}+\varepsilon$, it should either decrease very gently or jump to $\frac{1}{4}+\varepsilon$. Our main result shows that this is possible. The idea is to construct an information structure whose distribution over posteriors decays exponentially around the prior. This guarantees that agents' actions do not change their successors' beliefs too much, but at the same time also that beliefs cannot get stuck suboptimally.

Can the retailer always guarantee the optimal value? The next example demonstrates that she cannot.

Example 2. Suppose the agents have four actions, and so $A = \{a_1, a_2, a_3, a_4\}$. Action a_1 is optimal in the belief interval $\left[0, \frac{1}{4}\right]$, action a_2 is optimal in $\left[\frac{1}{4}, \frac{1}{2}\right]$, action a_3 is optimal in $\left[\frac{1}{2}, \frac{4}{5}\right]$, and action a_4 is optimal in $\left[\frac{4}{5}, 1\right]$. The payoffs to the retailer are $v(a_1) = 0$, $v(a_2) = 3$, $v(a_3) = 4$, and $v(a_4) = 1$. This is illustrated in Figure 1.

Now suppose towards a contradiction that for the prior $\mu = \frac{9}{10}$, the value of the concavification can be attained. This means that the public beliefs must converge either to the vicinity of $\mu_1 = \frac{4}{5}$ or to the vicinity of $\mu_2 = 1$. However, our main result shows that this is impossible. To see the intuition, observe that, in order for the public belief to approach 1, there must be some strong low signal. Otherwise, the public belief might converge to a number significantly lower than 1. However, in this case, the public belief will never end up close to $\frac{4}{5}$. Given any public belief in the interval $\left[\frac{1}{2}, \frac{4}{5}\right]$, the next agent will receive the aforementioned low signal with positive probability, in which case his belief will fall below $\frac{1}{2}$ and he will play action a_1 or a_2, leading the public belief to some value less than $\frac{1}{2}$. Thus, the public belief cannot converge to any value in interval $\left[\frac{1}{2}, \frac{4}{5}\right]$.

3 Model

Consider a standard herding model as formulated by Smith and Sørensen (2000). There is a binary state space $\Omega = \{0,1\}$ with a common prior μ that represents the prior probability of state $\omega = 1$, and a countable set of agents $N = \{1, 2, \ldots, \}$ indexed by the time parameter t. There is a finite set of actions $A = \{a_1, \ldots, a_\ell\}$, and a utility function $u : \Omega \times A \to \mathbb{R}$ common to all agents. One can write the optimal action for the agents with respect to u as a function of the belief $\lambda \in \Delta(\Omega) = [0,1]$. Agents are expected utility maximizers, and so the set of probabilities for which each action is optimal is a segment. We henceforth assume that action a_i is optimal for $\lambda \in [x_{i-1}, x_i]$, where $0 = x_0 < x_1 < x_2 < \cdots < x_\ell = 1$.[3] Finally, an information structure $G = (S, G_0, G_1)$ is comprised of a measurable space S and two probability measures $G_\omega \in \Delta(S)$ for $\omega \in \Omega$.

We often identify a posterior $\lambda \in [0,1]$ with its log-likelihood ratio $lr(\lambda) = \log\left(\frac{\lambda}{1-\lambda}\right)$. Thus, for $i = 1, \ldots, \ell$, the intervals $[x_{i-1}, x_i]$ in which action a_i is optimal are translated to $J_i = [y_{i-1}, y_i]$, where $y_i = lr(x_i) \in \mathbb{R} \cup \{-\infty, \infty\}$ for every $0 \le i \le \ell$. For any such interval, let $|J_i| = y_i - y_{i-1}$, and note that $|J_1| = |J_\ell| = \infty$. All other intervals are of finite length. Over these finite-length intervals we make the simplifying genericity assumption that $|J_i| \neq |J_k|$ whenever $i \neq k$, an assumption that allows us to avoid imposing tie-breaking restrictions on the agents' equilibrium behavior.

Given a herding model, the game is played as follows. At time $t = 0$ the unobserved state ω is realized according to μ. At each subsequent period t agent t observes the history of actions that were played by his predecessors, $h_t \in A^{t-1}$, receives a private signal s_t that is drawn independently according to G_ω, and chooses an action $a_t \in A$. A strategy

[3] This rules out the existence of distinct actions that yield the agents identical utilities in both states.

of agent t is a measurable mapping $\sigma_t : A^{t-1} \times S \to \Delta(A)$. As usual, a strategy profile $\sigma = (\sigma_t)_{t \in N}$ is a Bayesian Nash equilibrium if for every agent t the strategy σ_t maximizes his expected payoff given σ_{-t}. A strategy profile σ and an information structure G generate the *public belief* martingale $\{\mu_t\}_{t\in\mathbb{N}}$, where $\mu_t = \mathbf{P}_{G,\sigma}(\omega|h_t)$ represents the conditional probability of $\omega = 1$ given agent t's observed history. By the martingale convergence theorem the sequence $\{\mu_t\}_{t\in\mathbb{N}}$ converges almost surely to a limit μ_∞.

In addition, we consider a sender (designer, market maker) with a per-period utility $v : A \to \mathbb{R}$ from any action of the agent.[4] Given an information structure G and a corresponding equilibrium σ, the agents will eventually herd on some action (Smith and Sørensen, 2000). That is, from some finite time onward all agents will play the same action, say a^*. We identify the sender's utility as $V(G, \mu, \sigma) = E_{G,\sigma}[v(a^*)]$.[5]

We let $V(G, \mu)$ be the infimum of $V(G, \mu, \sigma)$ across all Bayesian equilibria σ of the herding game that correspond to information structure G and prior μ. We are interested in $V(\mu) = \sup_G V(G, \mu)$, the sender's optimal outcome, as well as the corresponding information structure with which this utility can be (almost) obtained.

We follow the Bayesian persuasion literature and define the indirect utility $v^* : \Delta(\Omega) \to \mathbb{R}$ as the expected utility of the sender evaluated at the (sender-optimal) best reply of the receiver. That is, $v^*(\lambda) = v(a(\lambda))$, where $a(\lambda)$ the best reply of the receiver, given belief λ, that is optimal for the sender.

Let $V^*(\mu)$ be the optimal value for a sender in a Bayesian persuasion game with a single receiver who has utility function u. By Aumann et al. (1995) and Kamenica and Gentzkow (2011), $V^*(\mu)$ is the *concavification* of v^*: It is the maximum value of $qv^*(\overline{\mu}) + (1-q)v^*(\underline{\mu})$ over all $\overline{\mu}$, $\underline{\mu}$, and $q \in [0, 1]$ that satisfy $\mu = q\overline{\mu} + (1-q)\underline{\mu}$ and $v^*(\overline{\mu}) \geq v^*(\underline{\mu})$. At the maximum, we say that $V^*(\mu)$ *is supported on* $\underline{\mu}$ and $\overline{\mu}$.[6] For $k \leq m$ let I_k and I_m be the intervals on which $\underline{\mu}$ and $\overline{\mu}$ lie, in some order. Specifically, if $\underline{\mu} \leq \overline{\mu}$ then $\underline{\mu} \in I_k$ and $\overline{\mu} \in I_m$, and, furthermore, since the sender's utility is state independent it must be that $\underline{\mu} = x_{k-1}$ and $\overline{\mu} = x_{m-1}$. Conversely, if $\underline{\mu} \geq \overline{\mu}$ then $\overline{\mu} \in I_k$ and $\underline{\mu} \in I_m$, and specifically $\underline{\mu} = x_m$ and $\overline{\mu} = x_k$.

Now, it is easy to see that $V^*(\mu) \geq V(\mu)$ for every prior μ. Our main question is,

[4] We discuss state-dependent utilities in Section 5.

[5] An equivalent definition is $V(G, \mu, \sigma) = \lim_{\delta\to 1}(1-\delta)\sum_{t\in N}\delta^{t-1}E_{G,\sigma}v(a_t)$. For a discussion regarding constant $\delta < 1$, see Section 5.

[6] $(\underline{\mu}, \overline{\mu})$ are not necessarily unique.

under what conditions does equality hold?

4 Main Result

In this section we state and discuss our main result, and then turn to its proof.

Theorem 1. $V(\mu) = V^*(\mu)$ if and only if $V^*(\mu)$ is supported on beliefs $\underline{\mu}$ and $\overline{\mu}$ for which the following hold:

- If $\underline{\mu} \in I_k$, $\overline{\mu} \in I_m$, and $\underline{\mu} \leq \overline{\mu}$, then $|J_m| \geq |J_k| > |J_i|$ for every integer $i \in (k, m)$.[7]
- If $\overline{\mu} \in I_k$, $\underline{\mu} \in I_m$, and $\underline{\mu} \geq \overline{\mu}$, then $|J_k| \geq |J_m| > |J_i|$ for every integer $i \in (k, m)$.

To understand the logic behind Theorem 1, suppose $\underline{\mu} \leq \overline{\mu}$, and note that in order to approximate $V^*(\mu)$ the limit public belief martingale must converge to small neighborhoods of $\underline{\mu}$ and $\overline{\mu}$. The condition $|J_k| > |J_i|$ guarantees the existence of a signal distribution for which the public belief "skips" intermediate intervals and gets absorbed in either I_m or I_k.

Furthermore, when $\underline{\mu} \leq \overline{\mu}$ we have $\underline{\mu} = x_{k-1}$, the farthest point from the prior μ on I_k. In contrast, $\overline{\mu} = x_{m-1}$, the closest point to μ in I_m. This means that the public belief martingale should be able to cross the interval I_k, but should never cross I_m. This can be achieved when $|J_m| \geq |J_k|$.

We next outline the proof of the first direction of Theorem 1. Given a pair of posteriors $\underline{\beta} \in I_k$ and $\overline{\beta} \in I_m$, Proposition 1 provides a sufficient condition over the information structure F such that the limits of the public beliefs μ_∞ lie arbitrarily close to $\underline{\beta}$ and $\overline{\beta}$ for each equilibrium strategy. We prove this by characterizing which posteriors $\lambda \in [0, 1]$ are "fixed points" of the public-belief martingale (Lemmas 1 and 2) and by providing a novel signal construction that yields $\underline{\beta}$ and $\overline{\beta}$ as limiting beliefs (Lemmas 3 and 4). We then show that the pair $(\underline{\mu}, \overline{\mu})$ satisfies the condition of Proposition 1.

A simple case in which the conditions of Theorem 1 are satisfied is when there are only two possible actions, as in Example 1. Another simple case is when the pair $(a_k, a_m) = (a_1, a_\ell)$—namely, the beliefs on which the concavification is supported correspond to the respective optimal actions in the two states—since then the corresponding intervals satisfy $|J_k| = |J_m| = \infty$.

[7] By convention, if $|J_k| = \infty$ and $|J_m| = \infty$ then $|J_k| = |J_m|$.

4.1 Proof of Theorem 1

Consider an information structure $G = (S, G_0, G_1)$. As standard in the herding literature, we identify G with a measure $F \in \Delta([0,1])$ that has expectation $\frac{1}{2}$, and that represents the posterior distribution G generates for the prior $\frac{1}{2}$. More precisely, let $G_{\frac{1}{2}} \in \Delta(\Omega \times S)$ be the measure that is induced by G and the prior $\mu = \frac{1}{2}$ over $\Omega \times S$. Let $p(s) = G_{\frac{1}{2}}(\omega = 1|s)$ be the conditional probability of state $\omega = 1$ given the signal $s \in S$. For every Borel measurable subset $B \subseteq [0,1]$ let

$$F(B) = G_{\frac{1}{2}}(p(s) \in B).$$

By the splitting lemma of Aumann et al. (1995), the identification goes in both directions: Namely, every $F \in \Delta([0,1])$ with an expectation of $\frac{1}{2}$ defines an information structure with $S = [0,1]$. In this case it holds that $p(s) = s$ almost surely. We also note that any such information structure F defines two probability measures, $F_0, F_1 \in \Delta([0,1])$, each of which represents the conditional posterior distribution given state ω and prior $\frac{1}{2}$.[8]

Under this identification, we make use of the fact that if the public belief at time t is μ_t and agent t receives a signal s_t, then, by Bayes' rule, his posterior probability p_t of state $\omega = 1$ satisfies $\frac{p_t}{1-p_t} = \frac{\mu_t}{1-\mu_t}\frac{s_t}{1-s_t}$. Therefore, in particular, $lr(p_t) = lr(\mu_t) + lr(s_t)$.

For clarity of the proof we make a simplifying genericity assumption that if $V^*(\mu) > v^*(\mu)$, then $V^*(\mu)$ is supported on exactly two points.

Definition 1. Consider a pair of posterior beliefs $\underline{\beta}, \overline{\beta} \in [0,1]$ such that $\underline{\beta} < \mu < \overline{\beta}$. Such a pair is *feasible for* μ if for every $\varepsilon > 0$ there exists an information structure F such that, for any equilibrium of the herding game, it holds with probability 1 that $\mu_\infty \in (\underline{\beta} - \varepsilon, \underline{\beta} + \varepsilon) \cup (\overline{\beta} - \varepsilon, \overline{\beta} + \varepsilon)$.

We will use the following auxiliary result.

Proposition 1. Consider a pair $0 < \underline{\beta} < \mu < \overline{\beta} < 1$ and let $\underline{\eta} = lr(\underline{\beta})$ and $\overline{\eta} = lr(\overline{\beta})$. Assume $\underline{\eta} \in int(J_k)$ and $\overline{\eta} \in int(J_m)$, where $1 \leq k < m \leq \ell$. If the following conditions hold, then the pair is feasible for μ:

1. $|J_i| < y_k - \underline{\eta} + \overline{\eta} - y_{m-1}$ for every integer $i \in (k, m)$;
2. $y_k - \underline{\eta} < y_m - \overline{\eta}$ and $\overline{\eta} - y_{m-1} < \underline{\eta} - y_{k-1}$.

[8]For more details see, e.g., Acemoglu et al. (2011) and Arieli et al. (2020).

To prove Proposition 1 we will require several definitions and lemmas. For an information structure $F \in \Delta([0,1])$, let $\underline{\alpha}_F = \inf\{x|F([0,x]) > 0\}$ and $\overline{\alpha}_F = \sup\{x|F([0,x]) < 1\}$ be its strongest negative and positive signals, respectively. The logic behind the proof of Proposition 1 is as follows. If we take an information structure F with $lr(\underline{\alpha}_F) = y_{m-1} - \overline{\eta}$ and $lr(\overline{\alpha}_F) = y_k - \underline{\eta}$, then the limit of the public belief $\mu_\infty \notin (\underline{\beta}, \overline{\beta})$ with probability 1. This follows from the first condition of the proposition, which implies that $\mu_\infty \notin I_i$ for every $i \in (k, m)$ (Lemma 1). Moreover, we will show that if μ_∞ reaches either a small left neighborhood of $\underline{\beta}$ or a small right neighborhood $\overline{\beta}$, then it gets absorbed (Lemma 2). The challenge is to construct an information structure F for which, in every equilibrium, the public belief $\{\mu_t\}_{t\in\mathbb{N}}$ indeed reaches these neighborhoods. In Lemmas 3 and 4 we construct such a structure.

We make use of the following definition.

Definition 2. Given an information structure F, belief $\lambda \in (0,1)$ is a *cascade point* if, whenever the game starts with λ as a prior, it holds that $\mu_t = \lambda$ for every t in every equilibrium σ of the herding game. Conversely, λ is a *continuation point* if there exists $\delta > 0$ such that, for every equilibrium and every initial prior, $\mu_\infty \notin (\lambda - \delta, \lambda + \delta)$ with probability 1.

The following two lemmas characterize the cascade and continuation points of any information structure F. The first is straightforward and its proof is omitted.

Lemma 1. Let F be an information structure with $\overline{\alpha}_F = \overline{\alpha}$ and $\underline{\alpha}_F = \underline{\alpha}$. Let $[x, y] \subseteq (x_{i-1}, x_i)$ for some $1 \leq i \leq \ell$. If either $lr(y) + lr(\underline{\alpha}) < y_{i-1}$ or $lr(x) + lr(\overline{\alpha}) > y_i$, then all points in $[x, y]$ are continuation points. Moreover, if $y_i - y_{i-1} < lr(\overline{\alpha}) - lr(\underline{\alpha})$, then all points in $[x_{i-1}, x_i]$ are continuation points. Conversely, if $\lambda \in [x_{i-1}, x_i]$ such that $lr(\lambda) + lr(\underline{\alpha}) > y_{i-1}$ and $lr(\lambda) + lr(\overline{\alpha}) < y_i$, then λ is a cascade point.

Lemma 2. Consider a pair $0 < \underline{\beta} < \mu < \overline{\beta} < 1$ that satisfies the conditions of Proposition 1. Let $\underline{\varphi} = y_{m-1} - \overline{\eta}$ and $\overline{\varphi} = y_k - \underline{\eta}$. If F is any information structure with $\underline{\alpha} = lr^{-1}(\underline{\varphi}) = \frac{\exp(\underline{\varphi})}{1+\exp(\underline{\varphi})}$ and $\overline{\alpha} = lr^{-1}(\overline{\varphi}) = \frac{\exp(\overline{\varphi})}{1+\exp(\overline{\varphi})}$, then there exists $\delta > 0$ such that all points in $(\underline{\beta} - \delta, \underline{\beta}) \cup (\overline{\beta}, \overline{\beta} + \delta)$ are cascade points. In addition, all points in $(\overline{\beta}, \underline{\beta})$ are continuation points.

Proof. Note first that since $-\infty < \underline{\varphi} < 0$ and $0 < \overline{\varphi} < \infty$, we have that $0 < \underline{\alpha} < \frac{1}{2} <$

$\overline{\alpha} < 1$. Also, by definition, $\overline{\eta} + \underline{\varphi} = y_{m-1}$. Furthermore, by condition 2 of Proposition 1,

$$\overline{\eta} + \overline{\varphi} = \overline{\eta} + y_k - \underline{\eta} < \overline{\eta} + y_m - \overline{\eta} = y_m.$$

Hence, for a sufficiently small constant $r > 0$, every $\psi \in (\overline{\eta}, \overline{\eta} + r)$ satisfies $\psi + \underline{\varphi} > y_{m-1}$ and $\psi + \overline{\varphi} < y_m$. Lemma 1 implies that $lr^{-1}(\psi)$ is a cascade point. Thus, for a sufficiently small $\delta > 0$, any $\lambda \in (\overline{\beta}, \overline{\beta} + \delta)$ is a cascade point. The fact that, for an appropriate choice of $\delta > 0$, every $\lambda \in (\underline{\beta} - \delta, \underline{\beta})$ is a cascade point, is shown similarly.

We next show that all points in $(\overline{\beta}, \underline{\beta})$ are continuation points. Consider a point $\psi \in [y_{i-1}, y_i]$ for some integer $i \in (k, m)$. By the first condition of Proposition 1 we have that $\overline{\varphi} - \underline{\varphi} > y_i - y_{i-1}$. Therefore, Lemma 1 implies that in any equilibrium of the game $\mu_\infty \notin [x_{i-1}, x_i]$ with probability 1, and all points in $[x_{i-1}, x_i]$ are continuation points.

Furthermore, if $\psi \in [a, b] \subseteq (y_{m-1}, \overline{\eta})$, then by definition $\psi + \underline{\varphi} < y_{m-1}$. Thus, Lemma 1 implies that $lr(\mu_\infty) \notin (y_{m-1}, \overline{\eta})$ with probability 1. Similar considerations imply that $lr(\mu_\infty) \notin (\underline{\eta}, y_k)$ with probability 1. $\square$

The following lemma shows that we can construct an information structure in which the likelihood ratio of the posterior does not move far from the prior.

Lemma 3. For every $0 \leq \underline{\alpha} < \frac{1}{2} < \overline{\alpha} \leq 1$ and $\delta > 0$ there exists an information structure F with $\underline{\alpha}_F = \underline{\alpha}$ and $\overline{\alpha}_F = \overline{\alpha}$ that satisfies the following conditions:

1. If $\frac{1}{2} \leq x < y \leq \overline{\alpha}$ and $F([x, y]) > 0$, then $|\frac{F_1([x,y])}{F_0([x,y])} - \frac{x}{1-x}| \leq \delta$,

2. if $\underline{\alpha} \leq x < y \leq \frac{1}{2}$ and $F([x, y]) > 0$, then $|\frac{F_1([x,y])}{F_0([x,y])} - \frac{y}{1-y}| \leq \delta$,

3. if $\underline{\alpha} \leq x \leq \frac{1}{2} \leq y \leq \overline{\alpha}$ and $F([x, y]) > 0$, then $|\frac{F_1([x,y])}{F_0([x,y])} - 1| \leq \delta$.

The main idea is to construct an information structure in which the posterior distribution decays exponentially on either side of the prior.

Proof. Let $\beta = \frac{1/2 - \underline{\alpha}}{\overline{\alpha} - \underline{\alpha}}$. We first define an information structure as a function of two parameters, $b \in (0, 1)$ and $n \in \mathbb{N}$. For every $n > 0$ we define a sequence of $2n$ numbers, as follows. For $j \in [1, n]$, $z_j = 1/2 + j \cdot \frac{\overline{\alpha} - 1/2}{n}$ and $z_{j+n} = \frac{1/2 - \beta z_j}{1 - \beta}$. Note first that, by definition, $z_n = \overline{\alpha}$ and $z_{2n} = \underline{\alpha}$. Second, as n grows, the sequence $Z_n = \{z_1, \ldots, z_{2n}\}$ gets more dense in the interval $[\underline{\alpha}, \overline{\alpha}]$, namely, $\max_{x \in [\underline{\alpha}, \overline{\alpha}]} \min_{j \in [1, 2n]} |z_j - x| \to 0$ as $n \to \infty$. Finally, for $j \in [1, n]$ it holds that $z_j > 1/2$, $z_{n+j} < 1/2$, and $\beta z_j + (1 - \beta) z_{n+j} = 1/2$.

We next define an information structure $F \in \Delta(Z_n)$ over the sequence of points. Fix the value $b > 0$. For any $j \in [1, n]$, let $F(z_j) = \beta b^j / C$ and $F(z_{n+j}) = (1-\beta)b^j/C$, where C is a normalizing constant. Since $\frac{1}{F(z_j)+F(z_{j+n})}[F(z_j)z_j + F(z_{j+n})z_{n+j}] = 1/2$, the distribution F has expectation $1/2$, as required.

Fix some $z \in Z_n$. Lemma 1 in Arieli et al. (2020) implies that the conditional probability of z given state $\omega = 1$ is $F_1(z) = 2zF(z)$, and its conditional probability given state $\omega = 0$ is $F_0(z) = 2(1-z)F(z)$.

Consider condition 1 in the statement of the lemma, and let $\frac{1}{2} \leq x < y \leq \overline{\alpha}$, where $F([x, y]) > 0$. Let j be the minimal index in $\{1, \ldots, n\}$ such that $x \leq z_j$ and let i be the maximal index in $\{1, \ldots, n\}$ such that $z_i \leq y$. Note that $F[x, y] = F[z_j, z_i]$, and, by the above,

$$\frac{F_1([z_j, z_i])}{F_0([z_j, z_i])} = \frac{\sum_{k\in[j,i]} z_k b^k}{\sum_{k\in[j,i]}(1-z_k)b^k} = \frac{z_j + b\sum_{k\in[j+1,i]} z_k b^{k-j-1}}{(1-z_j) + b\sum_{k\in[j+1,i]}(1-z_k)b^{k-j-1}}.$$

The expression goes to $\frac{z_j}{1-z_j}$ as b goes to zero. In addition, since $|z_j - x| \to 0$ as $n \to \infty$ it holds that $\frac{z_j}{1-z_j} \to \frac{x}{1-x}$ as $n \to \infty$. Therefore, if n is sufficiently large, then one can choose a sufficiently small $b > 0$ for which $|\frac{F_1([x,y])}{F_0([x,y])} - \frac{x}{1-x}| \leq \delta$. Condition 2 follows similarly.

Turning to condition 3, note that if $F([x, y]) > 0$ and $\frac{1}{2} \in [x, y]$, then either z_1 or z_n (or both) lies in $[x, y]$. Note that both $\frac{F(z_j)}{F(z_1)}$ and $\frac{F(z_j)}{F(z_n)}$ approach 0 as $b \to 0$ for every $j \neq 1, n$. Therefore, $\frac{F_1([x,y])}{F_0([x,y])}$ approaches $\frac{F_1(z_1)\mathbf{1}_{[x,y]}(z_1)+F_1(z_n)\mathbf{1}_{[x,y]}(z_n)}{F_0(z_1)\mathbf{1}_{[x,y]}(z_1)+F_0(z_n)\mathbf{1}_{[x,y]}(z_n)}$ as $b \to 0$, where $\mathbf{1}_{[x,y]}$ is the indicator function for $[x, y]$. In addition, since z_1 and z_n approach $\frac{1}{2}$ as n goes to infinity, it follows from the above definition of F_0 and F_1 that $\frac{F_1(z_1)}{F_0(z_1)}$ and $\frac{F_1(z_n)}{F_0(z_n)}$ approach 1 as $n \to \infty$. Together, these imply that condition 3 holds for sufficiently large n and sufficiently small b. $\square$

The next lemma shows that we can construct an information structure in which the public belief martingale does not change too quickly.

Lemma 4. Consider a herding game. For every ε and $\underline{\alpha} < \frac{1}{2} < \overline{\alpha}$ there exists an information structure F with $\underline{\alpha}_F = \underline{\alpha}$ and $\overline{\alpha}_F = \overline{\alpha}$ such that, in any equilibrium σ, the following conditions on the public belief martingale $\{\mu_t\}_t$ hold. For any time t and $1 \leq j \leq \ell$, with probability 1,

1. if $\mu_t \in [x_{j-1}, x_j]$ and $a_t = a_r$ for $r > j$ then $|\mu_{t+1} - x_{r-1}| \leq \varepsilon$,

2. if $\mu_t \in [x_{j-1}, x_j]$ and $a_t = a_r$ for $r < j$ then $|\mu_{t+1} - x_r| \leq \varepsilon$, and

3. if $\mu_t \in [x_{j-1}, x_j]$ and $a_t = a_j$ then $|\mu_{t+1} - \mu_t| \leq \varepsilon$.

To prove the lemma, we show that the information structure constructed in Lemma 3 satisfies the conditions of Lemma 4.

Proof. Fix $\delta > 0$, and let F be the information structure guaranteed by Lemma 3. Consider the first case, where $\mu_t \in [x_{j-1}, x_j]$ and $a_t = a_r$ for $r > j$. Since σ is an equilibrium, it follows that the posterior distribution s_t of player t, conditional on the realized history h_t and his private signal, satisfies

$$\frac{\mu_t}{1-\mu_t} \cdot \frac{s_t}{1-s_t} \in \left[\frac{x_{r-1}}{1-x_{r-1}}, \frac{x_r}{1-x_r}\right].$$

Hence

$$\frac{s_t}{1-s_t} \in \left[\frac{1-\mu_t}{\mu_t} \cdot \frac{x_{r-1}}{1-x_{r-1}}, \frac{1-\mu_t}{\mu_t} \cdot \frac{x_r}{1-x_r}\right].$$

Define $z_{r-1} \in [0,1]$ so that $\frac{z_{r-1}}{1-z_{r-1}} = \frac{1-\mu_t}{\mu_t} \cdot \frac{x_{r-1}}{1-x_{r-1}}$, and z_r so that $\frac{z_r}{1-z_r} = \frac{1-\mu_t}{\mu_t} \cdot \frac{x_r}{1-x_r}$. We note that since $\mu_t < x_{r-1}$ we must have that $\frac{1}{2} < z_{r-1}$. By Bayes' rule it follows that

$$\frac{\mu_{t+1}}{1-\mu_{t+1}} = \frac{\mu_t}{1-\mu_t} \cdot \frac{F_1([z_{r-1}, z_r])}{F_0([z_{r-1}, z_r])}.$$

By the assumption on F and by condition 1 in Lemma 3 we have that $\left|\frac{F_1([z_{r-1},z_r])}{F_0([z_{r-1},z_r])} - \frac{z_{r-1}}{1-z_{r-1}}\right| \leq \delta$. By the definition of z_r we must have that $\frac{\mu_{t+1}}{1-\mu_{t+1}} \to \frac{x_{r-1}}{1-x_{r-1}}$ as $\delta \to 0$. Hence, for sufficiently small δ it holds that $|\mu_{t+1} - x_{r-1}| \leq \varepsilon$. The other two cases follow similarly. □

We now prove Proposition 1.

Proof. Assume that the two conditions are satisfied for $\underline{\beta}$ and $\overline{\beta}$. Let $\delta > 0$ be the value guaranteed by Lemma 2 and let $\underline{\varphi} = y_{m-1} - \overline{\eta}$, $\overline{\varphi} = y_k - \underline{\eta}$, $\underline{\alpha} = lr^{-1}(\underline{\varphi})$, and $\overline{\alpha} = lr^{-1}(\overline{\varphi})$.

Let F be the information structure guaranteed by Lemma 4, with $\underline{\alpha}_F = \underline{\alpha}$, $\overline{\alpha}_F = \overline{\alpha}$ and $2\varepsilon < \delta$. Note that $\overline{\varphi} - \underline{\varphi} = y_k - \underline{\eta} + \overline{\eta} - y_{m-1}$. By assumption, $\overline{\varphi} - \underline{\varphi} > |J_i|$ for every integer $i \in (k, m)$. Therefore, Lemma 2 implies that, in every equilibrium of the corresponding game, $\mu_\infty \in [0, x_k) \cup (x_{m-1}, 1]$ with probability 1.

We claim that, for a sufficiently small $\varepsilon > 0$ and any equilibrium σ, if the public belief $\mu_t \leq x_{m-1}$, then $\mu_{t+1} < \overline{\beta}$. To see this, note that, by the choice of $\overline{\varphi}$, if $\lambda \in [0,1]$ satisfies $lr(\lambda) \leq y_{m-1}$, then

$$lr(\lambda) + \overline{\varphi} = lr(\lambda) + y_k - \underline{\eta} \leq y_{m-1} + y_m - \overline{\eta} < y_m.$$

This implies that, if the public belief $\mu_t \leq y_{m-1}$, then the posterior probability p_t of agent t after receiving his private signal satisfies $p_t < x_m$. Hence, with probability 1, agent t plays an action a_j such that $j \leq m$. Therefore, Lemma 4 implies that $\mu_{t+1} \leq x_{m-1} + \varepsilon$. Thus, for a sufficiently small $\varepsilon > 0$, we have $\mu_{t+1} \leq \overline{\beta}$.

Similarly, if $\mu_t \geq x_k$, then $\mu_t > \underline{\beta}$. Lemma 4 further guarantees that if $\mu_t \in \big(x_{m-1}, \overline{\beta}\big]$ and $a_t = a_m$, then $|\mu_{t+1} - \mu_t| \leq \varepsilon$. Similarly, if $\mu_t \in \big[\underline{\beta}, x_k\big)$ and $a_t = a_k$, then $|\mu_{t+1} - \mu_t| \leq \varepsilon$. This, together with the fact that all points $\lambda \in (\overline{\beta}, \underline{\beta})$ are continuation points, implies that μ_t reaches $(\underline{\beta} - \varepsilon, \underline{\beta}) \cup (\overline{\beta}, \overline{\beta} + \varepsilon)$ with probability 1. Since $\varepsilon < \delta$ whenever μ_t reaches $(\underline{\beta} - \varepsilon, \underline{\beta}) \cup (\overline{\beta}, \overline{\beta} + \varepsilon)$, at that point the martingale stops. Therefore, $\mu_\infty \in (\underline{\beta} - \varepsilon, \underline{\beta}) \cup (\overline{\beta}, \overline{\beta} + \varepsilon)$ with probability one for every equilibrium σ. This implies that the pair $0 \leq \underline{\beta} < \mu < \overline{\beta} \leq 1$ is feasible.

□

We now prove Theorem 1.

Proof. We first show, using Proposition 1, that the conditions of Theorem 1 are sufficient. We begin with the case where $\mu \in (x_{q-1}, x_q)$ and $V^*(\mu) = v^*(\mu)$. In this case, it is optimal not to reveal any information. Assume next that $\mu = x_{q-1}$, where we can assume that $q > 1$. Let ε be sufficiently small such that $\overline{\varphi}_\varepsilon = x_{q-1} + \varepsilon < x_q$. Let $\underline{\varphi} \in (x_{q-2}, x_{q-1})$ be such that $y_{q-1} - lr(\underline{\varphi}) < y_q - y_{q-1}$. By Proposition 1, the pair $(\underline{\varphi}, \overline{\varphi}_\varepsilon)$ is feasible for all sufficiently small ε. Let F_ε be the information structure that guaranteed by Proposition 1, for which $\mu_\infty \in (\underline{\varphi} - \varepsilon/2, \underline{\varphi} + \varepsilon/2) \cup (\overline{\varphi}_\varepsilon - \varepsilon/2, \overline{\varphi}_\varepsilon + \varepsilon/2)$ with probability 1 in any equilibrium σ. Note that $E_\sigma[\mu_\infty] = \mu$, that the distance of μ from $(\underline{\varphi} - \varepsilon/2, \underline{\varphi} + \varepsilon/2)$ is bounded away from zero, and that the distance of μ from $(\overline{\varphi}_\varepsilon - \varepsilon/2, \overline{\varphi}_\varepsilon + \varepsilon/2)$ approaches zero when $\varepsilon \to 0$. Therefore, when $\varepsilon \to 0$ the probability that μ_∞ lies in $(\overline{\varphi}_\varepsilon - \varepsilon/2, \overline{\varphi}_\varepsilon + \varepsilon/2)$ approaches 1. This implies that the expected utility of the sender lies arbitrarily close to $V^*(\mu)$. Similar considerations can be applied when $\mu = x_q$. This concludes the first case.

Assume next that $V^*(\mu) > v^*(\mu)$. Under this assumption there exist two priors $\underline{\mu}, \overline{\mu} \in [0, 1]$ with $v^*(\overline{\mu}) \geq v^*(\underline{\mu})$ and

$$V^*(\mu) = qv^*(\overline{\mu}) + (1 - q)v^*(\underline{\mu}),$$

where $q \in [0, 1]$. Assume that $\underline{\mu} < \mu < \overline{\mu}$ (the converse case where $\overline{\mu} < \mu < \underline{\mu}$ is shown analogously). Let $\underline{\mu} \in [x_{k-1}, x_k]$ and $\overline{\mu} \in [x_{m-1}, x_m]$. Since v^* is piecewise-constant with $v^*(\mu) < V^*(\mu)$ we can assume that $\underline{\mu} = x_{k-1}$ and $\overline{\mu} = x_{m-1}$.

Let us assume for now that $0 < x_{k-1} = \underline{\mu}$. By assumption, $|J_k| \leq |J_m|$ and $|J_k| > |J_i|$ for every integer $i \in (k, m)$. Let $\underline{\eta}^\delta = y_{k-1}+2\delta$ and $\overline{\eta}^\delta = y_{m-1}+\delta$. Let $x^\delta_{k-1} = lr^{-1}(\underline{\eta}^\delta)$ and $x^\delta_{m-1} = lr^{-1}(\overline{\eta}^\delta)$. We show that the two conditions of Proposition 1 hold, and so the pair $(x^\delta_{k-1}, x^\delta_{m-1})$ is feasible for all sufficiently small $\delta > 0$. The first condition follows since $y_k - \underline{\eta}^\delta + \overline{\eta}^\delta - y_{m-1}$ approaches $|J_k|$ and hence the inequality $y_k - \underline{\eta}^\delta + \overline{\eta}^\delta - y_{m-1} > |J_i|$ holds for every integer $i \in (k, m)$ and all sufficiently small $\delta > 0$. The second condition readily follows from the definition of $(x^\delta_{k-1}, x^\delta_{m-1})$ and the fact that $|J_k| \leq |J_m|$. Therefore, $(x^\delta_{k-1}, x^\delta_{m-1})$ is feasible for all sufficiently small $\delta > 0$ as desired.

By assumption, for every $\varepsilon > 0$ there exists F such that $\mu_\infty \in (x^\delta_{k-1} - \varepsilon, x^\delta_{k-1} + \varepsilon) \cup (x^\delta_{m-1} - \varepsilon, x^\delta_{m-1} + \varepsilon)$ with probability 1 in every equilibrium of the herding game. Note that $(x^\delta_{k-1}, x^\delta_{m-1})$ approaches (x_{k-1}, x_{m-1}) as δ goes to 0. Therefore, since $E[\mu_\infty] = \mu$, we must have that

$$\mathbf{P}_\sigma\Big(\mu_\infty \in (x^\delta_{k-1} - \varepsilon, x^\delta_{k-1} + \varepsilon)\Big) \to 1 - q$$

and

$$\mathbf{P}_\sigma\Big(\mu_\infty \in (x^\delta_{m-1} - \varepsilon, x^\delta_{m-1} + \varepsilon)\Big) \to q$$

as δ and ε go to zero. Thus, the population herds on a_k with probability approaching $1-q$ and a_m with probability approaching q. This approximates the Bayesian persuasion solution to any desirable precision.

The case where $x_{k-1} = 0$ is shown similarly, by observing that $(\delta, x_{m-1}+\delta)$ is feasible for all sufficiently small $\delta > 0$.

We next show that the converse holds. Again, we assume that $\underline{\mu} < \mu < \overline{\mu}$. We will show that if either $|J_i| > |J_k|$ for some integer $i \in (k, m)$ or $|J_m| < |J_k|$, then $V^*(\mu) > V(\mu)$.

Assume first that $|J_i| > |J_k|$ for some $i \in (k, m)$. Note that, for an information structure F, in order for the event $\mu_\infty \in (x_{m-1} - \varepsilon, x_{m-1} + \varepsilon)$ to hold with positive probability in equilibrium σ we must have that $\underline{\alpha}_F$ approaches $\frac{1}{2}$ with ε. Alternatively, for every $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that if $\underline{\alpha}_F \leq \frac{1}{2} - \varepsilon$, then $V(\mu) \leq V^*(\mu) - \delta(\varepsilon)$. Let $\varepsilon_0 = 1/2 - lr^{-1}(-\frac{|J_i|-|J_k|}{2})$. Note that since $|J_i| - |J_k| > 0$ it holds that $\epsilon_0 > 0$.

We consider two cases. If F is such that $\underline{\alpha}_F \leq \frac{1}{2} - \varepsilon_0$, then by the above we have that $V(\mu) \leq V^*(\mu) - \delta(\varepsilon_0)$. Otherwise, $\underline{\alpha}_F > \frac{1}{2} - \varepsilon_0$ and $lr(\underline{\alpha}_F) \geq -\frac{|J_i|-|J_k|}{2}$. Note that $lr(\overline{\alpha}_F) < |J_k|$, for otherwise, by Lemma 1, all points in $[x_{k-1}, x_k]$ are continuation points, and we would have that $\mu_\infty \notin [x_{k-1}, x_k]$ with probability 1. Thus, in this case, the sender's utility must be bounded away from $V^*(\mu)$.

Since $lr(\underline{\alpha}_F) \geq -\frac{|J_i|-|J_k|}{2}$ and $lr(\overline{\alpha}_F) < |J_k|$ it follows from Lemma 1 that any point $\lambda \in [x_{i-1}, x_i]$ with $lr(\lambda) \in [y_{i-1} + \frac{|J_i|-|J_k|}{2}, y_i - |J_k|]$ is a cascade point. Hence, in any equilibrium, if the public belief reaches a point μ_t such that $lr(\mu_t) \in [y_{i-1} + \frac{|J_i|-|J_k|}{2}, y_i - |J_k|]$, then learning stops and $\mu_\infty = \mu_t$. Thus, whenever μ_t satisfies $lr(\mu_t) > y_{i-1} + \frac{|J_i|-|J_k|}{2}$, it cannot down-cross $lr^{-1}[y_{i-1} + \frac{|J_i|-|J_k|}{2}]$ and reach $[x_{k-1}, x_k]$. If $\mu_t \in [x_{i-1}, x_i]$, then it holds with positive probability that $\mu_\infty \in [x_{i-1}, x_i]$. This demonstrates that $\mu_\infty \in [x_{i-1}, x_i]$ with positive probability. Hence, the sender's utility is bounded away from $V^*(\mu)$.

We next show that if $|J_k| > |J_m|$, then $V(\mu) < V^*(\mu)$. As before, for every $\varepsilon > 0$ there exists a $\delta(\varepsilon)$ such that if $\underline{\alpha}_F \leq \frac{1}{2} - \varepsilon$, then $V(\mu) \leq V^*(\mu) - \delta(\varepsilon)$. Let $\varepsilon_0 = 1/2 - lr^{-1}(-\frac{|J_k|-|J_m|}{4})$. If F is such that $\underline{\alpha}_F \leq \frac{1}{2} - \varepsilon_0$, then by the above we have that $V(\mu) \leq V^*(\mu) - \delta(\varepsilon_0)$. Otherwise, $\underline{\alpha}_F > \frac{1}{2} - \varepsilon_0/2$ and $lr(\underline{\alpha}_F) \geq -\frac{|J_k|-|J_m|}{4}$. Note that $lr(\overline{\alpha}_F) < |J_m|$, for otherwise, by Lemma 1, we would have that $\mu_\infty \notin [x_{m-1}, x_m]$ with probability 1, and as a result the sender's utility is bounded away from $V^*(\mu)$. It follows from Lemma 1 that any point $\lambda \in [x_{k-1}, x_k]$ with $lr(\lambda) \in [y_{k-1} + \frac{|J_i|-|J_k|}{4}, y_k - |J_m|]$ is a cascade point.

Thus, if the public belief reaches some μ_t with $lr(\mu_t) \in [y_{k-1} + \frac{|J_i|-|J_k|}{4}, y_k - |J_m|]$, then learning stops and $\mu_\infty = \mu_t$. This implies that if $\mu_{t'} \geq y_k - |J_m|$ for some time t', then $\mu_t \geq y_k - |J_m| - \frac{|J_m|-|J_k|}{4}$ at every subsequent time $t \geq t'$, since $lr(\underline{\alpha}_F) \geq -\frac{|J_k|-|J_m|}{4}$. Therefore, in this case the sender's utility is also bounded away from $V^*(\mu)$. $\square$

5 Discussion

We analyzed the classic herding model with an endogenously chosen information structure, and identified necessary and sufficient conditions under which the sender can manipulate the crowd as well as she can manipulate any individual.

In our model, we assumed that the sender's utility is $V(G, \mu, \sigma) = E_{G,\sigma}[v(a^*)] = \lim_{\delta \to 1}(1-\delta)\sum_{t \in N} \delta^{t-1} E_{G,\sigma} v(a_t)$. For the case in which the utility function is not evaluated at the limit of $\delta \to 1$, but rather at some fixed $\delta \in (0, 1)$, we have the following observation: $V(\mu) = V^*(\mu)$ if and only if one of the following holds: (i) either $V^*(\mu) = v^*(\mu)$, in which case it is optimal for the sender to not reveal any information; or (ii) $V^*(\mu)$ is supported on $\underline{\mu} = 0$ and $\overline{\mu} = 1$, in which case it is optimal for the sender to fully reveal the state. The reason is simple: When $\delta \in (0, 1)$, the only way for the sender to obtain utility $V^*(\mu)$ is if she extracts this utility from every agent separately.

Although we focused on a sender with a state-independent utility function, our arguments extend to the case in which this utility is state dependent, namely $v : \Omega \times A \to \mathbb{R}$. In this case, the indirect utility v^* is no longer piecewise-constant, but it does remain piecewise-linear. The necessary and sufficient conditions under which $V(\mu) = V^*(\mu)$ will then be determined by whether the beliefs $\underline{\mu}$ and $\overline{\mu}$, which support $V^*(\mu)$, are on the left or right endpoints of the respective intervals J_k and J_m on which they lie: If the supporting beliefs are on the left endpoints or on the right endpoints of both respective intervals, then the conditions and arguments are nearly identical to the state-independent case. In contrast, if the supporting beliefs are on different endpoints of their respective intervals, then $V(\mu) = V^*(\mu)$ if and only if $\underline{\mu} = 0$ and $\overline{\mu} = 1$.

The intuition is the following. If $\underline{\mu} = 0$ and $\overline{\mu} = 1$, then $V^*(\mu)$ can be attained by choosing the information structure that fully reveals the state. On the other hand, suppose that $\underline{\mu} > 0$, $\overline{\mu} < 1$, $J_k < J_m$, and that $\underline{\mu}$ lies on the left endpoint of J_k and $\overline{\mu}$ on the right endpoint of J_m. In this case, in order for the public belief to get sufficiently close to $\underline{\mu}$, there must be a sufficiently strong positive signal s. However, this signal will prevent the public belief from getting close to $\overline{\mu}$: whenever the belief starts to approach $\overline{\mu}$ from the left, an agent who obtains signal s will take an action a_r for $r > m$, thereby "overshooting" past the desired belief $\overline{\mu}$. The other cases suffer from the same problem – further details appear in the Online Appendix.

Finally, we conjecture that our result applies to more general observation structures (as in Lobel and Sadler, 2015), but leave this to future research.

References

Acemoglu, D., Dahleh, M. A., Lobel, I. and Ozdaglar, A. (2011). Bayesian learning in social networks. *The Review of Economic Studies*, **78** 1201–1236.

Arieli, I., Babichenko, Y. and Smorodinsky, R. (2020). Identifiable information structures. *Games and Economic Behavior*, **120** 16–27.

Arieli, I., Sandomirskiy, F. and Smorodinsky, R. (2021). On social networks that support learning. *Available at SSRN 3728165*.

Au, P. H. (2015). Dynamic information disclosure. *The RAND Journal of Economics*, **46** 791–823.

Aumann, R. J., Maschler, M. and Stearns, R. E. (1995). *Repeated games with incomplete information*. MIT press.

Bahar, G., Arieli, I., Smorodinsky, R. and Tennenholtz, M. (2020). Multi-issue social learning. *Mathematical Social Sciences*, **104** 29–39.

Banerjee, A. V. (1992). A simple model of herd behavior. *Quarterly Journal of Economics*, **107** 797–817.

Bikhchandani, S., Hirshleifer, D. and Welch, I. (1992). A theory of fads, fashion, custom, and cultural change as informational cascades. *Journal of Political Economy*, **100** 992–1026.

Candogan, O. (2020). On information design with spillovers. *Available at SSRN 3537289*.

Chen, Y., Wang, Q. and Xie, J. (2011). Online social interactions: A natural experiment on word of mouth versus observational learning. *Journal of marketing research*, **48** 238–254.

Duan, W., Gu, B. and Whinston, A. B. (2009). Informational cascades and software adoption on the internet: an empirical investigation. *MIS Quarterly* 23–48.

Egorov, G. and Sonin, K. (2020). Persuasion on networks. Tech. rep., National Bureau of Economic Research.

Ely, J. C. (2017). Beeps. *American Economic Review*, **107** 31–53.

Galperti, S. and Perego, J. (2020). Information systems. Tech. rep.

Kamenica, E. and Gentzkow, M. (2011). Bayesian persuasion. *American Economic Review*, **101** 2590–2615.

Li, F. and Norman, P. (2021). Sequential persuasion. *Theoretical Economics*, **16** 639–675.

Lobel, I. and Sadler, E. (2015). Information diffusion in networks through social learning. *Theoretical Economics*, **10** 807–851.

Lorecchio, C. (2022). Persuading crowds. *UB Economics–Working Papers, 2022, E22/434*.

Lorecchio, C. and Monte, D. (forthcoming). Dynamic information design under constrained communication rules. *American Economic Journal: Microeconomics*.

Sgroi, D. (2002). Optimizing information in the herd: Guinea pigs, profits, and welfare. *Games and Economic Behavior*, **39** 137–166.

Smith, L. and Sørensen, P. (2000). Pathological outcomes of observational learning. *Econometrica*, **68** 371–398.

Smith, L., Sørensen, P. and Tian, J. (2020). Informational herding, optimal experimentation, and contrarianism. *The Review of Economic Studies*.

Thies, F., Wessel, M. and Benlian, A. (2016). Effects of social interaction dynamics on platforms. *Journal of Management Information Systems*, **33** 843–873.

Tucker, C. and Zhang, J. (2011). How does popularity information affect choices? a field experiment. *Management Science*, **57** 828–842.

Welch, I. (1992). Sequential sales, learning, and cascades. *The Journal of Finance*, **47** 695–732.

Zhang, J. and Liu, P. (2012). Rational herding in microloan markets. *Management Science*, **58** 892–912.

Online Appendix

In this online appendix we consider the more general model in which the sender's utility is state-dependent, and so the per-period utility is $v : A \times \Omega \to \mathbb{R}$. As in the state-independent case, for any prior μ such that $V^*(\mu) > v^*(\mu)$ the function $V^*(\mu)$ is supported on two priors, $\underline{\mu} \in [0,1]$ and $\overline{\mu} \in [0,1]$. Note that the function v^* is piece-wise linear,[9] as is its concavification V^*.

Now, consider a prior $\mu \in [x_{q-1}, x_q]$. For every $\lambda \in [0,1]$ and action $a \in A$ let $v(\lambda, a) = \lambda v(1,a) + (1-\lambda) v(0,a)$. The piece-wise linearity of V^* implies that one of the following five cases must hold:

Case 1: $V^*(\mu) = v^*(\mu)$.

Case 2: $(\mu, V^*(\mu))$ lies on the line joining $(x_{k-1}, v(x_k, a_k))$ with $(x_{m-1}, v(x_{m-1}, a_m))$ for some $k \leq q \leq m$.

Case 3: $(\mu, V^*(\mu))$ lies on the line joining $(x_k, v(x_k, a_k))$ with $(x_m, v(x_m, a_m))$ for some $k \leq q \leq m$.

Case 4: $(\mu, V^*(\mu))$ lies on the line joining $(x_{k-1}, v(x_{k-1}, a_k))$ with $(x_m, v(x_m, a_m))$ for some $k \leq q \leq m$.

Case 5: $(\mu, V^*(\mu))$ lies on the line joining $(x_k, v(x_k, a_k))$ with $(x_{m-1}, v(x_{m-1} a_m))$ for some $k \leq q \leq m$.

The necessary and sufficient conditions under which $V(\mu) = V^*(\mu)$ are determined by whether the beliefs $\underline{\mu}$ and $\overline{\mu}$, which support $V^*(\mu)$, are on the left or right endpoints of the respective intervals J_k and J_m on which they lie. In particular, if the supporting beliefs are on the left endpoints or on the right endpoints of both respective intervals, then the conditions and arguments are nearly identical to the state-independent case. In contrast, if the supporting beliefs are on different endpoints of their respective intervals, then $V(\mu) = V^*(\mu)$ if and only if $\underline{\mu} = 0$ and $\overline{\mu} = 1$ (in which case full revelation is optimal). The following theorem tightly characterizes the conditions under which $V(\mu) = V^*(\mu)$.

[9] To see this, recall that in any interval of beliefs $[x_{i-i}, x_i]$ for $i \in \{1, \ldots, \ell\}$, the agents' optimal action is a_i. For any $\lambda \in [x_{i-i}, x_i]$, the sender's indirect utility is thus the linear $v^*(\lambda) = \lambda \cdot v(1, a_i) + (1-\lambda) \cdot v(0, a_i)$.

Theorem 2. $V^*(\mu) = V(\mu)$ if and only if one of the following conditions is true:

- Case 1 holds.
- Case 2 holds and $|J_k| \geq |J_m|$ and $|J_m| \geq |J_i|$ for every integer $i \in (k, m)$.
- Case 3 holds and $|J_m| \geq |J_k|$ and $|J_k| \geq |J_i|$ for every integer $i \in (k, m)$.
- Case 4 holds with $k - 1 = 0$ and $m = l$.

The intuition for the proof of Theorem 2 is the following. If $\underline{\mu} = 0$ and $\overline{\mu} = 1$, then $V^*(\mu)$ can be attained by choosing the information structure that fully reveals the state. On the other hand, suppose that $\underline{\mu} > 0$, $\overline{\mu} < 1$, and $J_k < J_m$, and that $\underline{\mu}$ lies on the left endpoint of J_k and $\overline{\mu}$ on the right endpoint of J_m. In this case, in order for the public belief to get sufficiently close to $\underline{\mu}$, there must be a sufficiently strong positive signal s. However, this signal will prevent the public belief from getting close to $\overline{\mu}$: whenever the belief starts to approach $\overline{\mu}$ from the left, an agent who obtains signal s will take an action a_r for $r > m$, thereby "overshooting" past the desired belief $\overline{\mu}$. The other cases, such as $\underline{\mu}$ lying on the right endpoint of J_k and $\overline{\mu}$ on the left endpoint of J_m, suffer from the same problem.

We first adapt the proof of Proposition 1 to apply also to state-dependent utility.

Proof. Assume that the two conditions are satisfied for $\underline{\beta}, \overline{\beta}$. Let $\delta > 0$ be the value guaranteed by Lemma 2 and let $\underline{\varphi} = y_{m-1} - \overline{\eta}$, $\overline{\varphi} = y_k - \underline{\eta}$, $\underline{\alpha} = lr^{-1}(\underline{\varphi})$, and $\overline{\alpha} = lr^{-1}(\overline{\varphi})$.

Let F be an information structure that is guaranteed in Lemma 4 such that $\underline{\alpha}_F = \underline{\alpha}$, $\overline{\alpha}_F = \overline{\alpha}$ and $2\epsilon < \delta$. We note that $\overline{\varphi} - \underline{\varphi} = y_k - \underline{\eta} + \overline{\eta} - y_{m-1}$. By assumption $\overline{\varphi} - \underline{\varphi} > |J_i|$ for every integer $i \in (k, m)$. Therefore, Lemma 2 implies that for every equilibrium of the corresponding game, $\mu_\infty \in [0, x_k) \cup (x_{m-1}, 1]$ with probability 1.

We claim that for a sufficiently small $\epsilon > 0$ it holds for any equilibrium σ that if the public belief $\mu_t \leq x_{m-1}$, then $\mu_{t+1} < \overline{\beta}$. To see this note that by the choice of $\overline{\varphi}$ it holds that if $\lambda \in [0, 1]$ such that $lr(\lambda) \leq y_{m-1}$, then

$$lr(\lambda) + \overline{\varphi} = lr(\lambda) + y_k - \underline{\eta} \leq y_{m-1} + y_m - \overline{\eta} < y_m.$$

Hence if the public belief $\mu_t \leq y_{m-1}$, then the posterior probability p_t of agent t after receiving his private signal satisfies $p_t < x_m$. Hence with probability one agent t plays an action a_j such that $j \leq m$. Therefore Lemma 4 implies that $\mu_{t+1} \leq x_{m-1} + \epsilon$. Thus for a sufficiently small $\epsilon > 0$ we have $\mu_{t+1} \leq \overline{\beta}$.

Similarly, if $\mu_t \geq x_k$, then $\mu_t \geq \underline{\beta}$. Lemma 4 further guarantees that if $\mu_t \in (x_{m-1}, \overline{\beta}]$ and $a_t = a_m$, then $|\mu_{t+1} - \mu_t| \leq \epsilon$. Similarly if $\mu_t \in [\underline{\beta}, x_{m-1})$ and $a_t = a_k$, then $|\mu_{t+1} - \mu_t| \leq \epsilon$. This, together with the fact that all points $\lambda \in (\overline{\beta}, \underline{\beta})$ are continuation points, implies that μ_t reaches $(\underline{\beta} - \epsilon, \underline{\beta}) \cup (\overline{\beta}, \overline{\beta} + \epsilon)$ with probability one. Since $\epsilon < \delta$ whenever μ_t reaches $(\underline{\beta} - \epsilon, \underline{\beta}) \cup (\overline{\beta}, \overline{\beta} + \epsilon)$ the martingale stops. Therefore we must have that $\mu_\infty \in (\underline{\beta} - \epsilon, \underline{\beta}) \cup (\overline{\beta}, \overline{\beta} + \epsilon)$ for every equilibrium σ as desired. Hence the pair $0 \leq \underline{\beta} < \mu < \overline{\beta} \leq 1$ is feasible.

□

We now prove Theorem 2.

Proof. We first show, using Proposition 1, that the conditions of Theorem 1 are sufficient. We begin with the case where $\mu \in (x_{q-1}, x_q)$ and $V^*(\mu) = v^*(\mu)$. In this case, it is optimal not to reveal any information. Assume next that $\mu = x_{q-1}$, where we can assume that $q > 1$. Let ε be sufficiently small such that $\overline{\varphi}_\varepsilon = x_{q-1} + \varepsilon < x_q$. Let $\underline{\varphi} \in (x_{q-2}, x_{q-1})$ be such that $y_{q-1} - lr(\underline{\varphi}) < y_q - y_{q-1}$. By Proposition 1, the pair $(\underline{\varphi}, \overline{\varphi}_\varepsilon)$ is feasible for all sufficiently small ε. Let F_ε be the information structure that guaranteed by Proposition 1, for which $\mu_\infty \in (\underline{\varphi} - \varepsilon/2, \underline{\varphi} + \varepsilon/2) \cup (\overline{\varphi}_\varepsilon - \varepsilon/2, \overline{\varphi}_\varepsilon + \varepsilon/2)$ with probability 1 in any equilibrium σ. Note that $E_\sigma[\mu_\infty] = \mu$, that the distance of μ from $(\underline{\varphi} - \varepsilon/2, \underline{\varphi} + \varepsilon/2)$ is bounded away from zero, and that the distance of μ from $(\overline{\varphi}_\varepsilon - \varepsilon/2, \overline{\varphi}_\varepsilon + \varepsilon/2)$ approaches zero when $\varepsilon \to 0$. Therefore, when $\varepsilon \to 0$ the probability that μ_∞ lies in $(\overline{\varphi}_\varepsilon - \varepsilon/2, \overline{\varphi}_\varepsilon + \varepsilon/2)$ approaches 1. This implies that the expected utility of the sender lies arbitrarily close to $V^*(\mu)$. Similar considerations can be applied when $\mu = x_q$. This concludes case 1.

If case 4 holds, the fully revealing information structure is optimal for the sender.

Assume that case 2 holds (case 3 is similar). We start with considering the case where $0 < x_{k-1}$ and $|J_k| > |J_m| > |J_i|$ for every integer $i \in (k, m)$. Let $\underline{\eta}^\delta = lr(x_{k-1}) + \delta$ and $\overline{\eta}^\delta = lr(x_{m-1}) + \delta$. Let $x^\delta_{k-1} = lr^{-1}(\underline{\eta}^\delta)$ and $x^\delta_{m-1} = lr^{-1}(\overline{\eta}^\delta)$. We claim that the pair $(x^\delta_{k-1}, x^\delta_{m-1})$ is feasible for all sufficiently small $\delta > 0$. To see this we show that the two conditions of Proposition 1 hold. The second condition holds since $y_k - \underline{\eta}^\delta + \overline{\eta}^\delta - y_{m-1}$ approaches $|J_k|$ as $\delta \to 0$. Similarly, the first condition holds for all sufficiently small

$\delta > 0$ since $|J_m| > |J_k|$ and since $\overline{\eta}^\delta - y_{m-1} = \underline{\eta}^\delta - y_{k-1} = \delta$.

Therefore, $(x^\delta_{k-1}, x^\delta_{m-1})$ is feasible for all sufficiently small $\delta > 0$. By assumption, for every $\epsilon > 0$ there exists F such that $\mu_\infty \in (x^\delta_{k-1} - \epsilon, x^\delta_{k-1} + \epsilon) \cup (x^\delta_{m-1} - \epsilon, x^\delta_{m-1} + \epsilon)$ with probability one in every equilibrium of the herding game. Note that $(x^\delta_{k-1}, x^\delta_{m-1})$ approaches (x_{k-1}, x_{m-1}) as δ goes to 0. Therefore since $E[\mu_\infty] = \mu$ we must have that $\mathbf{P}_\sigma\Big(\mu_\infty \in (x^\delta_{k-1} - \epsilon, x^\delta_{k-1} + \epsilon)\Big)$ approaches $\frac{x_{m-1}-\mu}{x_{m-1}-x_{k-1}}$ and $\mathbf{P}_\sigma\Big(\mu_\infty \in (x^\delta_{m-1} - \epsilon, x^\delta_{m-1} + \epsilon)\Big)$ approaches $\frac{\mu - x_{k-1}}{x_{m-1}-x_{k-1}}$ as δ and ϵ go to zero. This means that the actions on which the population cascades are a_k with probability approaching $\frac{x_{m-1}-\mu}{x_{m-1}-x_{k-1}}$ and a_m with probability approaching $\frac{\mu - x_{k-1}}{x_{m-1}-x_{k-1}}$. This approximates the Bayesian persuasion solution to any desired precision.

The case where $x_{k-1} = 0$ is shown similarly, by observing that $(\delta, x_{m-1} + \delta)$ is feasible for all sufficiently small $\delta > 0$.

We next show that the converse hold. Namely, we start with case 2 and show that if it holds and either $|J_i| > |J_m|$ for some integer $i \in (k, m)$ or $|J_m| > |J_k|$, then $V^*(\mu) > V(\mu)$ (the converse for case 3 is shown similarly).

Assume first that $|J_i| > |J_m|$ for some integer $i \in (k, m)$. Note that for an information structure F, in order for the event $\mu_\infty \in (x_{m-1} - \epsilon, x_{m-1} + \epsilon)$ to hold with positive probability in equilibrium σ we must have that $\underline{\alpha}_F$ approaches $\frac{1}{2}$ with ϵ. Alternatively, for every $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that if $\underline{\alpha}_F \leq \frac{1}{2} - \epsilon$, then $V(\mu) \leq V^*(\mu) - \delta(\varepsilon)$. Let $\epsilon_0 = 1/2 - lr^{-1}(-\frac{|J_i| - |J_m|}{2})$.

We consider two cases. If F is such that $\underline{\alpha}_F \leq \frac{1}{2} - \epsilon_0$, then by the above we have that $V(\mu) \leq V^*(\mu) - \delta(\varepsilon_0)$. Otherwise, $\underline{\alpha}_F > \frac{1}{2} - \epsilon_0/2$ and $lr(\underline{\alpha}_F) \geq -\frac{|J_i| - |J_m|}{2}$. We note that $lr(\overline{\alpha}_F) \leq |J_m|$ for otherwise, by Lemma 1 we must have that $\mu_\infty \notin [x_{m-1}, x_m]$ with probability one and the utility for the is bounded away from $V^*(\mu)$. Hence, any point $\lambda \in [x_{i-1}, x_i]$ with $lr(\lambda) \in [y_{i-1} + \frac{|J_i| - |J_m|}{2}, y_i - |J_m|]$ is a cascade point. Thus, in any equilibrium, if the public belief reaches a point μ_t such that $lr(\mu_t) \in [y_{i-1} + \frac{|J_i| - |J_m|}{2}, y_i - |J_m|]$, then learning stops and $\mu_\infty = \mu_t$. Thus, whenever μ_t satisfies $lr(\mu_t) \geq y_{i-1} + \frac{|J_i| - |J_m|}{2}$, it cannot down-cross $lr^{-1}(y_{i-1} + \frac{|J_i| - |J_m|}{2})$ and reach $[x_{k-1}, x_k]$. Therefore if $\mu_t \in [x_{i-1}, x_i]$, then it holds with positive probability that $\mu_\infty \in [x_{i-1}, x_i]$. This demonstrates that $\mu_\infty \in [x_{i-1}, x_i]$ holds with positive probability. Hence the sender's utility is bounded away from $V^*(\mu)$.

We next show that if $|J_k| > |J_m|$, then $V(\mu) < V^*(\mu)$. As before, for every $\varepsilon > 0$

there exists a $\delta(\varepsilon)$ such that if $\underline{\alpha}_F \leq \frac{1}{2} - \epsilon$, then $V(\mu) \leq V^*(\mu) - \delta(\varepsilon)$. Let $\epsilon_0 = 1/2 - lr^{-1}(-\frac{|J_k|-|J_m|}{4})$. If F is such that $\underline{\alpha}_F \leq \frac{1}{2} - \epsilon_0$, then by the above we have that $V(\mu) \leq V^*(\mu) - \delta(\varepsilon_0)$. Otherwise, $\underline{\alpha}_F > \frac{1}{2} - \epsilon_0/2$ and $lr(\underline{\alpha}_F) \geq -\frac{|J_k|-|J_m|}{4}$. We note that $lr(\overline{\alpha}_F) \leq |J_m|$ for otherwise we would have that $\mu_\infty \notin [x_{m-1}, x_m]$ with probability one and the sender's utility will be bounded away from $V^*(\mu)$. Hence, any point $\lambda \in [x_{k-1}, x_k]$ with $lr(\lambda) \in [y_{k-1} + \frac{|J_i|-|J_k|}{4}, y_k - |J_m|]$ is a cascade point.

Thus, if the public belief reaches a point μ_t such that $lr(\mu_t) \in [y_{k-1}+\frac{|J_i|-|J_k|}{4}, y_k-|J_m|]$, then learning stops and $\mu_\infty = \mu_t$. This implies that if $\mu \geq y_k - |J_m|$, then $\mu_t \geq y_k - |J_m| - \frac{|J_m|-|J_k|}{4}$ for every t. This is true since $lr(\underline{\alpha}_F) \geq -\frac{|J_k|-|J_m|}{4}$. Hence, the sender's utility is bounded away from $V^*(\mu)$.

Finally, we show that if Case 4 holds and either $k-1 \neq 0$ or $m \neq l$, then $V(\mu) \neq V^*(\mu)$. We show this for the case $k-1 \neq 0$ ($m \neq l$ is similar). Let F be an information structure. If F is such that $\underline{\alpha}_F \leq \frac{1}{2} - \epsilon$, then, by Lemma 1, in any equilibrium σ the limit $\mu_\infty \notin [x_{k-1}, x_{k-1} + r)$, for sufficiently small $r > 0$, with probability one, and hence $V(\mu) \leq V^*(\mu) - \delta(\varepsilon_0)$. Similarly, if F is such that $\overline{\alpha}_F \geq \frac{1}{2} + \epsilon$, then by Lemma 1 it holds for sufficiently small $r > 0$ that in any equilibrium σ the limit $\mu_\infty \notin [x_m - r, x_m]$ with probability one and hence $V(\mu) \leq V^*(\mu) - \delta(\varepsilon_0)$. In contrast, if both $\underline{\alpha}_F \geq \frac{1}{2} - \epsilon$ and $\overline{\alpha}_F \leq \frac{1}{2} + \epsilon$, then for some constant $r(\epsilon) > 0$ all points $[x_{m-1} + r(\epsilon), x_m - r(\epsilon)]$ are cascade point. In addition, $r(\epsilon)$ goes to zero as ϵ goes to zero. This implies that for sufficiently small ϵ it holds that $\mu_\infty < x_m - r$ for some $r > 0$ and $V(\mu) < V^*(\mu)$. Hence in any case we have that $V(\mu) < V^*(\mu)$.

□